Comparison of Statistical Orbit Determination Methods

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Theme

THE results obtained in a comparison of the convergence characteristics and accuracy of the batch estimation algorithm and the extended sequential estimation algorithm, as applied to the problem of estimating the state of a near-earth satellite in the presence of geopotential modeling errors, are described. High-accuracy laser range observations of the Beacon Explorer-C satellite, during four consecutive passes, were used in the study. The results indicate that 1) when the solution is iterated, the batch algorithm converges to an estimate which differs from the extended sequential estimate by values less than the observation noise; 2) the extended sequential estimator achieves single iteration convergence whereas several iterations are required by the batch estimator; 3) the convergence rate of the batch algorithm is more dependent on the initial state error than the extended sequential algorithm; 4) the difference in estimates from the two methods produced a predicted position difference of less than one meter after three days; and 5) the radius of convergence of the extended sequential estimation algorithm appears to be about ten times larger than the radius of convergence for the batch estimator.

Contents

The problem of estimating the state of a nonlinear dynamical system using discrete observations which are corrupted by random observation error was first solved by Gauss[1] using the method of least squares. Since that time, considerable research has been devoted to the problem of improving the classical method of least squares and to placing the state estimation problem on a firm statistical foundation. For the linear estimation problem with Gaussian random errors, it has been shown that the estimation algorithm for the linear, unbiased, minimum variance estimator is identical to the weighted least squares if the weighting matrix is the observation noise covariance matrix.[2] Equivalence with a maximum likelihood estimator can also be shown. These methods will subsequently be referred to as "batch processors" since an entire batch of data is processed before an estimate of the state at some epoch is made. The form of the batch processor commonly used in orbit determination uses an a priori state error covariance matrix (for example, see Refs. 3 and 4). This form was used for the comparisons discussed in subsequent paragraphs. The sequential form of the estimator in which state estimates are obtained at each observation time is generally attributed to Kalman and Bucy.[5] Since the sequential form can be derived from the batch algorithm through algebraic manipulation, the two are mathematically equivalent. Furthermore, in both formulations, a fixed reference solution is used in processing all of the data. The extended sequential estimator or extended Kalman filter[6] differs from the Kalman-Bucy filter in that a fixed reference solution is not used. Instead, the reference solution is updated or rectified to the estimate of the state at each observation time. This rectification reduces the effects of nonlinearities on the estimator. Establishment of a formal mathematical equivalence between the extended sequential and the batch is difficult, but it is intuitively obvious that the converged results should be the same.

Because of the difficulties in establishing a rigorous proof of equivalence, one of the purposes of this investigation was to establish the level of numerical equivalence between the batch and extended sequential algorithms and to consider the convergence behavior of these algorithms. In particular, the study sought to determine the convergence rate and to ascertain whether differences exist in the radius of convergence.

The observations used in the study consisted of laser ranges of the Beacon Explorer C (BE-C) satellite obtained from two stations in 1970. One of the stations was a mobile facility located at Seneca Lake, N.Y., and the other was the fixed facility at the Goddard Space Flight Center.[7] The standard deviations of the range observations were typically 30 cm and 50 cm, respectively. Although data were collected for approximately six months, only the four consecutive passes of BE-C on September 2, 1970, were used in the comparison. For these four passes, a total of 3397 range observations were obtained from the two stations. The BE-C has an orbital period of 108 minutes, an eccentricity of 0.025, and an inclination of 41.2°. The data were processed by UTOPIA, an orbit determination program in use at The University of Texas[8] with capabilities for processing data in one of several programed estimation modes. The mathematical model consisted of the complete Smithsonian Standard Earth II geopotential,[9] luni-solar perturbations,[10] and polar motion.[11] The geopotential used is not capable of fitting the range observations to within their noise level; thus, the batch processor residuals exhibit periodic behavior due to the geopotential modeling problems with a typical root sum square (RSS) of two to four meters. During the first two passes, the extended sequential filter fits the data very well with a maximum RSS of 1.4 m. However, the model error produces larger residuals in the third pass of 7.1 m RSS. By the fourth pass, the batch and the sequential residuals are quite similar.

To compare results obtained using the batch processor with those obtained using the extended sequential processor, a common epoch must be used. With the batch processor, a state estimate is obtained at the initial time, whereas an estimate is obtained at each observation time with the sequential algorithm. To relate the estimates to a common epoch, the extended sequential state estimate at the end of a data arc was used as the initial condition for a backwards integration to the epoch at which the batch estimate is obtained. Furthermore, the estimates from the two processors are made using the same initial states and a priori state error covariance matrices. Finally, the same observational data were reprocessed by each estimator until the state estimate on two successive iterations differed by less than a specified constant. In all cases, the initial a priori covariance matrix was used to start each iteration; however, the reference orbit was defined by the estimate of the state at the initial epoch resulting from the previous iteration. Local iteration of the extended sequential algorithm was not used, i.e., the filter was applied only once at each observation time within the data arc.

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Three representative a priori state error cases were used in the comparisons between the batch and the extended sequential algorithms. These cases differed from the converged state as follows: Case 1a—approximately 80 m in position and 0.02 m/sec in velocity; Case 1b—approximately 80 m in position and 0.1 m/sec in velocity; and Case 2—approximately 900 m in position and 18 m/sec in velocity. The a priori state error covariance matrix was assumed to be diagonal with elements given by 400,000 m^2 and 150 m^2/sec^2 for the position and velocity components, respectively.

Comparison of the converged state estimates obtained by the batch and the extended sequential processors, reduced to a common epoch, reveal differences in the batch estimate relative to the extended sequential estimate as follows: $\Delta X = -5$ cm, $\Delta Y = 1$ cm, $\Delta Z = -11$ cm, $\Delta \dot{X} = -0.01$ mm/sec, $\Delta \dot{Y} = -0.1$ mm/sec, $\Delta \dot{Z} = 0.1$ mm/sec where the (X, Y, Z)-axes represent a geocentric nonrotating, equatorial coordinate system with X-axis oriented toward the vernal equinox of 1950.0. This comparison is based on processing the four passes of the BE-C satellite data and continuing the processor iteration until the position estimate on two successive iterations differed by less than 1 mm. The small differences in the batch and extended sequential estimates are the same for all three a priori state error cases. Furthermore, they are considerbly smaller than the assumed observation standard deviation of 1 meter. To further evaluate these differences, they were transformed into a tangential, normal, and binormal coordinate system defined by the extended sequential estimate. This resulted in position differences of -9 cm, -5 cm, and -5 cm for the four pass arc. The velocity components were 0.04 mm/sec, 0.14 mm/sec, and 0.05 mm/sec, respectively.

The behavior of the batch and extended sequential processors on each iteration is shown in Table 1. In this table, Δr represents the square root of the sum of the squares of the difference between the state estimate on a given iteration with the converged value. For all three a priori state errors, the extended sequential algorithm obtains an estimate of the state which differs from the converged value by less than 10 cm in the position on the first iteration. The batch algorithm, with the same a priori state error covariance, requires at least two iterations to achieve the accuracy obtained by the extended sequential algorithm on the first iteration. Furthermore, the number of iterations required by the batch algorithm for convergence is more dependent on the a priori state error than the extended sequential filter. For the Case 2 error, four iterations are required by the batch processor to achieve a position estimate accurate to within 10 cm compared to one iteration for the extended sequential processor.

The state error covariance matrix associated with the state estimates was compared also. As in the previous considerations, the comparison must be made at a common epoch. Thus, the covariance matrix associated with the state estimate of the extended sequential algorithm at the final observation was propagated backwards to the initial time for comparison with the covariance matrix associated with the batch estimated. The covariances at the common epoch indicate agreement to within 10^{-5} in position elements and 10^{-7} in velocity elements with units of meters and seconds.

Noting the small differences between the batch and extended sequential estimate described previously, a determination of how these differences propagate in a prediction mode is of interest. Using the two state estimates at the initial time, satellite state predictions to 400 min were made (equivalent to four passes). The state vectors of the two solutions were differenced at fixed intervals and the differences were transformed into a tangential, normal, and binormal coordinate system defined by the extended sequential orbit. The resulting time histories of the differences were approximated by secular and periodic functions. For the comparison of the converged batch and the single-iteration extended sequential, the position differences produced both secular and periodic terms in the tangential component and periodic terms in the normal and binormal components. The secular rate was found to be 1.36×10^{-4} m/min and the periodic terms had amplitudes comparable to the initial differences. A similar comparison between the converged batch orbit and the converged extended sequential orbit resulted in a secular rate of 2.70×10^{-4} m/min and periodic amplitudes comparable to the initial differences. The secular term will produce along-track differences between the batch estimate prediction and the extended sequential estimate prediction of less than 1 meter in three days.

To study the convergence properties further, each component of the converged batch estimate was multiplied by a factor representing a percentage error in the a priori state. The results obtained do not show absolute bounds for the radius of convergence; the intent was to ascertain whether any significant differences exist. Using the same a priori covariance matrix in both estimators, the experiments produced divergence in the batch algorithm at about 0.2% error in the initial state whereas the extended sequential algorithm diverged at 2.5%. The divergence conclusion was based on the fact that no apparent pattern toward convergence was discernible in 10 iterations. Finally, in all cases for which convergence was achieved, the extended sequential method required fewer iterations than the batch method.

Table 1 Comparison of iteration state estimates with converged estimates

Case	Iteration number	Batch Δr(m)	Batch Δv(m/sec)	Extended Sequential Δr(m)	Extended Sequential Δv(m/sec)
1a	Initial	80.00	0.0200	80.00	0.0200
	1	0.65	0.0019	0	0
	2	0	0		
1b	Initial	80.00	0.1000	80.00	0.1000
	1	6.76	0.0195	0	0
	2	0	0		
2	Initial	900.00	18.0000	900.00	18.0000
	1	18712.17	53.6575	0.08	0
	2	129.56	0.3972	0	0
	3	17.84	0.0512		
	4	0	0		

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